

# Spectral Factorization and Lattice Geometry

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## Abstract

We obtain conditions for a trigonometric polynomial  $t$  of one variable to equal or be approximated by  $|p|^2$  where  $p$  has frequencies in a Bohr set of integers obtained by projecting lattice points in the open planar region bounded by the lines  $y = \alpha x \pm \beta$  where  $|\beta| \leq \frac{1}{4}$  and  $\alpha$  is either rational or irrational with Liouville-Roth constant larger than 2. We derive and use a generalization of the Fejér-Riesz spectral factorization lemma in one dimension, an approximate spectral factorization in two dimensions, the modular group action on the integer lattice, and Diophantine approximation.

## 1 Introduction and Preliminary Results

We introduce notation and summarize standard results that we will use throughout the paper.  $\mathbb{Z}_+, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  denote the nonnegative integer, integer, rational, real, and complex numbers,  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  is the real circle group,  $\mathbb{T}_c = \{z \in \mathbb{C} : |z| = 1\}$  is the complex circle group, and the maps

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$e_j : \mathbb{T} \rightarrow \mathbb{T}_c, j \in \mathbb{Z}$  defined by  $e_j(x) = e^{2\pi i j x}$ ,  $x \in \mathbb{T}$  are homomorphisms. For  $z = x + iy, x, y \in \mathbb{R}$  we define  $\Re z = x$  and  $\Im z = y$ . If  $z \neq 0$  then there exist unique  $r > 0$  and  $-\pi < \theta \leq \pi$  such that  $z = re^{i\theta}$  and we define  $\log z = \log r + i\theta$  and  $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ .

The closed unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ .

For  $\sigma > 0$ ,  $\mathbb{A}_\sigma = \{z \in \mathbb{C} : e^{-\sigma} \leq |z| \leq e^\sigma\}$  is a closed annulus. For a topological space  $X$ ,  $C(X)$  denotes the Banach algebra of bounded continuous functions  $f : X \rightarrow \mathbb{C}$  with norm  $\|f\|_\infty = \sup \{|f(x)| : x \in X\}$ . For  $X \subset \mathbb{C}$ ,  $X^\circ$  denotes its interior and  $H(X)$  denotes the Banach subspace of  $C(X)$  consisting of functions whose restriction to  $X^\circ$  is holomorphic (complex analytic). For  $f \in C(\mathbb{T})$  we define  $\|f\|_1 = \int_{x \in \mathbb{T}} |f(x)| dx$ , the *Fourier transform*  $\hat{f} \in C(\mathbb{Z})$  by  $\hat{f}(j) = \int_{x \in \mathbb{T}} f(x) e_{-j}(x) dx$ ,  $j \in \mathbb{Z}$ , and the *frequency set*  $\text{freq}(f) = \{j \in \mathbb{Z} : \hat{f}(j) \neq 0\}$ . If  $f, h \in C(T)$  then  $\text{freq}(\bar{f}) = -\text{freq}(f)$  and  $\text{freq}(fh) \subseteq \text{freq}(f) + \text{freq}(h)$  and hence  $\text{freq}(|f|^2) \subseteq \text{freq}(f) - \text{freq}(f)$ . The convolution  $f * h$  of  $f, h \in C(T)$ , defined by  $(f * h)(x) = \int_{y \in \mathbb{T}} f(y) h(x - y) dx$ , satisfies  $\|f * h\|_\infty \leq \|f\|_1 \|h\|_\infty$  and  $\widehat{f * h} = \hat{f} \hat{h}$ . We let  $\ell^1(\mathbb{Z})$  denote the Banach space of absolutely summable  $f \in C(\mathbb{Z})$  with norm  $\|f\|_1 = \sum_{j \in \mathbb{Z}} |f(j)| < \infty$  and we let  $A(T)$  denote the *Wiener algebra* of  $f \in C(T)$  such that  $\hat{f} \in \ell^1(\mathbb{Z})$ . We let  $C^\omega(T)$  denote the algebra of *real analytic* functions on  $T$  and we let  $\mathfrak{T}_1$  denote the algebra of *trigonometric polynomials* that consists of  $f \in C(T)$  such that  $\text{freq}(f)$  is finite. For  $t \in \mathfrak{T}_1$  define  $n^-(t) = \min(\text{freq}(t))$ ,  $n^+(t) = \max(\text{freq}(t))$ ,  $n(t) = \max\{n^+(t), n^-(t)\}$ . For any  $F \subset \mathbb{Z}$ , we define  $\mathfrak{T}_1(F) = \{f \in \mathfrak{T}_1 : \text{freq}(f) \subseteq F\}$ . Clearly  $\mathfrak{T}_1 \subset C^\omega(T) \subset A(T) \subset C(T)$  and these algebras are closed under the involution  $f \rightarrow \bar{f}$ . We define the algebras  $A^\pm(\mathbb{T}) = \{f \in A(\mathbb{T}) : \text{freq}(f) \subseteq \pm\mathbb{Z}_+\}$  and observe that they are not closed under involution. For  $N \in \mathbb{Z}_+$  we define *Dirichlet*, *Hilbert* and *analytic* kernels by  $D_N = \sum_{j=-N}^N e_j$ ,  $H_N = \sum_{j=1}^N (e_j - e_{-j})$ , and  $A_N^\pm = \frac{1}{2}(D_N \pm H_N)$ . If  $f \in C(T)$  then  $D_N * f = \sum_{j=-N}^N \hat{f}(j) e_j$  and in general  $\|f - D_N * f\|_\infty$  does not converge to 0. If  $f \in A(T)$  then  $\|f - D_N * f\|_\infty \rightarrow 0$  as  $N$  increases and its rate of convergence increases with smoothness of  $f$ . The following approximation error bound holds [12, Chapter 10, Exercise 24].

**Lemma 1.1.**  *$f \in C^\omega(T)$  if and only if there exists  $\sigma > 0$  and  $F \in H(\mathbb{A}_\sigma)$  such that  $f = F(e_1)$ . Then for every  $N \geq 0$ ,*

$$\|f - D_N * f\|_\infty < 2 \|F\|_\infty \sigma^{-1} e^{-N\sigma}. \quad (1.1)$$

*Proof.* The first assertion follows since real analytic functions are closed under composition. If  $F \in H(\mathbb{A}_\sigma)$  then Cauchy's integral formula

$$F(z) = \frac{1}{2\pi i} \left( \int_{|w|=e^\sigma} - \int_{|w|=e^{-\sigma}} \right) \frac{F(w)}{w-z} dw, \quad z \in \mathbb{A}_\sigma^\circ,$$

implies that  $F$  has the Laurent expansion  $F(z) = \sum_{j \in \mathbb{Z}} c_j e^{-|j|\sigma} z^j$  where  $c_j = \int_0^1 F(e^\sigma e_1(x)) e_{-j}(x) dx$ ,  $j \geq 0$ ;  $c_j = \int_0^1 F(e^{-\sigma} e_1(x)) e_{-j}(x) dx$ ,  $j < 0$ . Therefore, if  $f = F(e_1)$  then  $\|f - D_N * f\|_\infty \leq 2 \|F\|_\infty e^{-(N+1)\sigma} / (1 - e^{-\sigma})$  and the second assertion follows since  $e^\sigma > 1 + \sigma$ .  $\square$

**Lemma 1.2.** *If  $f \in A^\pm(\mathbb{T})$  then  $\exp(f) \in A^\pm(\mathbb{T})$ . Furthermore, for any  $n, N \in \mathbb{Z}_+$  with  $n \leq N$ ,*

$$(1/2 + A_n^\pm) * \exp(f) = (1/2 + A_n^\pm) * \exp((1/2 + A_N^\pm) * f). \quad (1.2)$$

*Proof.* The first assertion follows since  $A(\mathbb{T})$  is a Banach algebra under the norm  $\|f\|_{A(\mathbb{T})} = \|\widehat{f}\|_1$ . Let  $g = (1/2 + A_N^\pm) * f$ . Since  $f - g = \sum_{j=N+1}^\infty \widehat{f}(j) e_{\pm j}$ ,  $\exp(f - g) = 1 + \sum_{j=N+1}^\infty c_j e_{\pm j}$  where  $c_j$  is absolutely summable. Therefore  $\exp(f) = \exp(g) + \sum_{j=N+1}^\infty d_j e_{\pm j}$  where  $d_j$  is absolutely summable and hence (1.2) holds.  $\square$

**Lemma 1.3.** *The kernels defined above satisfy the following inequalities.*

$$\|D_N\|_1 \leq 1 + \log(2N + 1). \quad (1.3)$$

$$\|A_N^\pm\|_1 \leq \frac{3}{2} + \log(N) \text{ and } \|1/2 + A_N^\pm\|_1 \leq 1 + \log(N + 1). \quad (1.4)$$

$$\|H_N\|_1 \leq 1 + 2 \log(N). \quad (1.5)$$

*Proof.* We reproduce the derivation of (1.3) given in [9, Section 16.2].

Since  $D_N^\pm(0) = 2N + 1$  and  $|D_N^\pm(x)| = \frac{\sin(\pi(2N+1)x)}{\sin \pi x}$ ,  $x \in (0, 1/2]$  it follows that  $|D_N(x)| \leq \min\{2N + 1, \frac{1}{2x}\}$ . Therefore for  $0 < u \leq \frac{1}{2}$ ,

$$\|D_N\|_1 \leq 2 \int_0^u (2N + 1) dx + 2 \int_u^{\frac{1}{2}} \frac{1}{2x} dx = 2(2N + 1)u - \log 2 - \log u.$$

Minimizing the function of  $u$  on the right gives (1.3). The inequalities in (1.4) follow from a similar argument and (1.5) follows since  $H_N = A_N^+ - A_N^-$ .  $\square$

For  $d \geq 1$ ,  $\mathfrak{T}_d$  is the algebra of trigonometric polynomials on  $\mathbb{T}^d$ , and  $\mathfrak{T}_d^+ = \{t \in \mathfrak{T}_d : t \geq 0\}$ . For  $F \subset \mathbb{Z}^d$ ,  $\mathfrak{T}_d(F) = \{t \in \mathfrak{T}_d : \text{freq}(t) \subseteq F\}$ ,  $\mathfrak{T}_d^+(F) = \mathfrak{T}_d^+ \cap \mathfrak{T}_d(F)$ ,  $\mathfrak{S}_d(F) = \{|p|^2 : p \in \mathfrak{T}_d(F)\}$ ,  $\mathfrak{U}_d(F) = \mathfrak{T}_d \cap \overline{\mathfrak{S}_d(F)}$ .

**Remark 1.4.** Clearly  $\mathfrak{S}_d(F) \subseteq \mathfrak{T}_d^+(F - F)$  but equality does not generally hold since  $t(x, y) = 5 + 2 \cos 2\pi x + 2 \cos 2\pi y$  is positive but irreducible. The Fejér-Riesz spectral factorization lemma ([11], p. 117), conjectured by Fejér [4] and proved by Riesz [10], shows that if  $F = \{0, \dots, n\}$  then  $\mathfrak{S}_d(F) = \mathfrak{T}_d^+(F - F)$ . If  $F$  is the set of lattice points in a half-space in  $\mathbb{R}^d$  then  $F - F = \mathbb{Z}^d$  and results of Helson and Lowdenslager [7] imply that  $\mathfrak{U}_d(F) = \mathfrak{T}_d^+$ . Rudin [13], Dritschel [3], Geronimo and Woerdneman [6], and others have extended these results.

For  $\alpha \in \mathbb{R}$  define  $\theta_\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$  and  $\Theta_\alpha : \mathfrak{T}_1 \rightarrow \mathfrak{T}_2$  by

$$\theta_\alpha(j) = \text{smallest integer that minimizes } |\theta_\alpha(j) - j\alpha|, \quad j \in \mathbb{Z}, \quad (1.6)$$

$$\Theta_\alpha(t)(x, y) = \sum_{j \in \mathbb{Z}} \widehat{t}(j) e_j(x) e_{\theta_\alpha(j)}(y), \quad t \in \mathfrak{T}_1. \quad (1.7)$$

For  $\alpha \in \mathbb{R}$  and  $\beta > 0$  define  $F_1(\alpha, \beta) \subseteq \mathbb{Z}$  and  $F_2(\alpha, \beta) \subseteq \mathbb{Z}^2$  by

$$F_1(\alpha, \beta) = \{j \in \mathbb{Z} : |\theta_\alpha(j) - j\alpha| < \beta\}, \quad (1.8)$$

$$F_2(\alpha, \beta) = \{(j, k) \in \mathbb{Z}^2 : |k - j\alpha| < \beta\}. \quad (1.9)$$

Clearly the map  $j \rightarrow (j, \theta_\alpha(j))$  maps  $F_1(\alpha, \beta)$  into  $F_2(\alpha, \beta)$  and projection onto the first coordinate is its left inverse.

**Remark 1.5.**  $F_1(\alpha, \beta)$  are *Bohr sets*, named after Harold Bohr whose work on almost periodic functions [1] they relate to. They arise in combinatorial number theory ([5], p. 132-137) and in symbolic dynamics since if  $\alpha$  is irrational and  $\beta \notin \alpha\mathbb{Z}$  then the characteristic function of  $F_1(\alpha, \beta)$  is a minimal sequence [8]. Their higher dimensional versions include quasicrystals such as the Penrose tilings discussed in [2].

**Lemma 1.6.** *If  $\alpha \in \mathbb{R}$ ,  $\frac{1}{4} \geq \beta > 0$  then  $\Theta_\alpha$  maps  $\mathfrak{T}_1(F_1(\alpha, 2\beta))$  bijectively onto  $\mathfrak{T}_2(F_2(\alpha, 2\beta))$ ,  $\Theta_\alpha(\mathfrak{S}_1(F_1(\alpha, \beta))) = \mathfrak{S}_2(F_2(\alpha, \beta))$ , and  $\Phi_\alpha(|p|^2) = |\Phi_\alpha(p)|^2$ ,  $p \in \mathfrak{T}_1(F_1(\alpha, \beta))$ .*

*Proof.* The first assertion follows since  $j \rightarrow (j, \theta_\alpha(j))$  gives a bijection of  $F_1(\alpha, 2\beta)$  onto  $F_2(\alpha, 2\beta)$  and  $\Theta_\alpha^{-1}(t)(x) = t(x, 0)$ ,  $t \in \mathfrak{T}_2(F_2(\alpha, 2\beta))$ , the second since  $\mathfrak{S}_1(F_1(\alpha, \beta)) \subseteq \mathfrak{T}_1(F_1(\alpha, 2\beta))$ , and the third since  $\theta_\alpha(m - n) = \theta_\alpha(m) - \theta_\alpha(n)$ ,  $m, n \in F_1(\alpha, \beta)$ .  $\square$

**Definition 1.7.**  $F \subset \mathbb{Z}^d$  has *Property A* if  $\mathfrak{U}_d(F) = \mathfrak{T}_d^+(F - F)$ .

**Corollary 1.8.** If  $\alpha \in \mathbb{R}$  and  $\frac{1}{4} \geq \beta > 0$  and  $F_2(\alpha, \beta)$  has *Property A* and  $t \in \mathfrak{T}_1(F_1(\alpha, \beta) - F_1(\alpha, \beta))$  and  $\Theta_\alpha(t) \in \mathfrak{T}_2^+$  then  $t \in \mathfrak{U}_1(F_1(\alpha, \beta))$ .

*Proof.* Under the assumptions there exists a sequence  $q_j \in \mathfrak{T}_2(F_2(\alpha, \beta))$  such that  $|q_j|^2 \rightarrow \Theta_\alpha(t)$ . Lemma 1.1 implies that there exists a sequence  $p_j \in \mathfrak{T}_1(F_1(\alpha, \beta))$  such that  $\Phi_\alpha(p_j) = q_j$  so  $\Phi_\alpha(|p_j|^2) = |q_j|^2 \rightarrow \Theta_\alpha(t)$  hence  $|p_j|^2 \rightarrow t$  so  $t \in \mathfrak{U}_1(F_1(\alpha, \beta))$ .  $\square$

## 2 Generalized Spectral Factorization

For  $f \in C(\mathbb{T})$  with  $|f| > 0$  we define the *winding number*

$$W(f) = \lim_{M \rightarrow \infty} \sum_{j=0}^{M-1} \log \left[ \frac{f((j+1)/M)}{f(j/M)} \right]. \quad (2.1)$$

For  $f \in C(\mathbb{T})$  with  $|f| > 0$  and  $W(f) = 0$  we define  $Lf \in C(\mathbb{T})$  by

$$(Lf)(x) = \log f(0) + \lim_{M \rightarrow \infty} \sum_{j=0}^{M-1} \log \left[ \frac{f((j+1)x/M)}{f(jx/M)} \right], \quad x \in \mathbb{T}. \quad (2.2)$$

For  $f \in A(\mathbb{T})$  we define  $A_\infty^\pm f = \frac{1}{2} \hat{f}(0) + \sum_{j=1}^\infty \hat{f}(\pm j) e_{\pm j}$ . Clearly  $W(fg) = W(f) + W(g)$ ,  $L(fg) = L(f) + L(g)$ ,  $\exp(Lf) = f$ , and if  $\Re f > 0$  then  $Lf = \log(f)$ . If  $f \in A(\mathbb{T})$  and  $|f| > 0$  and  $W(f) = 0$  then  $Lf \in A(\mathbb{T})$ .

**Definition 2.1.** For  $f \in A(\mathbb{T})$  with  $|f| > 0$  and  $W(f) = 0$  let

$$\Psi^\pm f = \exp(A_\infty^\pm(Lf)). \quad (2.3)$$

We record the following result whose proof is straightforward.

**Lemma 2.2.** If  $f \in A(\mathbb{T})$  with  $|f| > 0$  and  $W(f) = 0$  then  $\Psi^\pm f \in A^\pm(\mathbb{T})$  and  $(\Psi^+ f)(\Psi^- f) = f$ . Furthermore, if  $f > 0$  then  $\Psi^- f = \overline{\Psi^+ f}$ .

**Theorem 2.3.** *If  $t \in \mathfrak{T}_1$  and  $|t| > 0$  and  $W(t) = 0$  then there exist  $\lambda_1, \dots, \lambda_{n^-(t)}, \mu_1, \dots, \mu_{n^+(t)} \in \mathbb{D} \setminus \{0\}$  such that*

$$t = e^{\gamma(t)} \prod_{j=1}^{n^-(t)} (1 - \lambda_j e_{-1}) \prod_{j=1}^{n^+(t)} (1 - \mu_j e_1). \quad (2.4)$$

where  $\gamma(t) = \log t(0) - \sum_{j=1}^{n^-(t)} \log(1 - \lambda_j) - \sum_{j=1}^{n^+(t)} \log(1 - \mu_j)$ . Then

$$\Psi^+ t = e^{\gamma(t)/2} \prod_{j=1}^{n^+(t)} (1 - \mu_j e_1) \text{ and } \Psi^- t = e^{\gamma(t)/2} \prod_{j=1}^{n^-(t)} (1 - \lambda_j e_{-1}). \quad (2.5)$$

*Proof.* Since the polynomial  $P(z) = \sum_{j=-n^-(t)}^{n^+(t)} \widehat{t}(j) z^{n^-(t)+j}$ ,  $z \in \mathbb{C}$ , has degree  $n^-(t) + n^+(t)$ ,  $P(0) \neq 0$ , and  $P(e_1) = e_{n^-(t)} t$ , it has  $n^-(t) + n^+(t)$  roots, all roots  $\neq 0$ , and the modulus of every root  $\neq 1$ . Hence there exist integers  $m, n \geq 0$  with  $m + n = n^-(t) + n^+(t)$  and  $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \in \mathbb{D} \setminus \{0\}$  such that  $P(z) = \rho \prod_{j=1}^m (z - \lambda_j) \prod_{j=1}^n (1 - \mu_j z)$ . Hence  $W(P(e_1)) = m$ , and since  $P(e_1) = e_{n^-(t)} t$ ,  $W(P(e_1)) = W(e_{n^-(t)}) + W(t) = n^-(t)$ . Therefore  $m = n^-(t)$  so  $t = e_{-n^-(t)} P(e_1)$  which is (2.4) and (2.5) then follows.  $\square$

**Corollary 2.4.** *If  $t \in \mathfrak{T}_1$  and  $|t| > 0$  and  $W(t) = 0$  and  $N \geq n^\pm(t)$  then*

$$\|\Psi^\pm t\|_\infty \leq (1 + \log(n^\pm(t) + 1)) \exp \max(D_N * \log |t| + iH_N * \Im Lt)/2. \quad (2.6)$$

*Proof.* (1.2) and (2.3)  $\Rightarrow \Psi^\pm(t) = (\frac{1}{2} + A_{n^\pm(t)}^\pm) * \exp(A_N * (Lt))$ . Since  $D_N$  and  $iH_N$  are real,  $A_N = \frac{1}{2}(D_N \pm H_N)$ , and  $\Re Lt = \log |t|$ ,  $\|\exp A_N * (Lt)\|_\infty = \exp \max(D_N * \log |t| + iH_N * \Im Lt)/2$ . Therefore (2.6) follows from (1.4).  $\square$

**Remark 2.5.** For odd  $n \in \mathbb{Z}_+$  we let  $\ell = (n - 1)/2$  and construct the polynomial  $P_n(z) = (z - 1/n)^{2n} - 1$  and the trigonometric polynomial  $t_n = e_{-n} P_n(e_1)$ . Since the roots of  $P_n$  inside  $\mathbb{D}^\circ$  are  $-e^{\pi i k/n} + 1/n, k = -\ell, \dots, \ell$  and the roots of  $P_n$  outside  $\mathbb{D}$  are  $e^{\pi i k/n} + 1/n, k = -\ell, \dots, \ell$  and  $P_n(1/n) = 1$ , a direct computation shows that  $n^\pm(t) = n$  and

$$\Psi^- t_n = \kappa \prod_{k=-\ell}^{\ell} [1 - (-e^{\pi i k/n} + 1/n) e_{-1}], \quad \Psi^+ t_n = \kappa \prod_{k=-\ell}^{\ell} [1 - (e^{\pi i k/n} + 1/n)^{-1} e_1]$$

where  $\kappa = \sqrt{1 + 1/n} \prod_{j=1}^{\ell} |e^{\pi i j/n} + 1/n| \approx \sqrt{e}$ . For large  $n$ ,  $\log |t_n| \in [-2n, 3]$  yet  $\|\Psi^\pm t_n\|_\infty \approx 2^n$ . The large norm of the spectral factor originates from the fact that  $\|\Im Lt_n\|_\infty \approx \pi n/2$ . If  $\Re t > 0$  then  $\Im Lt = \Im \log t \in (-\pi/2, \pi/2)$  and we will obtain smaller bounds for  $\|\Psi^\pm t\|_\infty$ .

**Definition 2.6.** For  $t \in \mathfrak{T}_1$  that satisfies  $\Re t > 0$  let

$$\rho(t) = \min\{1, \min \Re t / (2e \|\widehat{t}\|_1)\}, \quad \sigma(t) = \rho(t)/n(t), \quad (2.7)$$

$$\tau(t) = \max\{\log(\|t\|_\infty + \min \Re t / 2), \pi/2 + |\log(\min \Re t / 2)|\}. \quad (2.8)$$

**Lemma 2.7.** If  $t \in \mathfrak{T}_1$  and  $\Re t > 0$  then there exists  $F \in H(\mathbb{A}_{\sigma(t)})$  such that  $\log t = F(e_1)$ . Furthermore,  $\|F\|_\infty \leq \tau(t)$ .

*Proof.* Let  $F = \log P$  where  $P$  is the Laurent polynomial such that  $t = P(e_1)$ . If  $z \in \mathbb{A}_{\sigma(t)}$  then  $z = e^s e_1(x)$  where  $|s| \leq \sigma(t)$  and  $x \in \mathbb{T}$ . Therefore  $|P(z) - t(x)| \leq \min \Re t / 2$  hence  $\Re P \geq \min \Re t / 2$  so  $F = \log P \in H(\mathbb{A}_{\sigma(t)})$  and  $\log t = F(e_1)$  which proves the first assertion. The second assertion follows from the triangle inequality and (2.8).  $\square$

**Theorem 2.8.** There exists a function  $\lambda : (0, \infty)^2 \rightarrow (0, \infty)$  such that

$$\|\Psi^\pm t\|_\infty \leq \lambda(\rho(t), \tau(t)) n(t)^{\pi/2}, \quad t \in \mathfrak{T}_1, \quad \Re t > 0. \quad (2.9)$$

*Proof.* Let  $\xi(N) = \max(D_N * \log |t| + iH_N * \Im Lt)/2$  and  $q = \max |\Im t|$ . Since  $\Re t > 0$ ,  $Lt = \log t$  so  $q < \pi/2$  and (1.1), (1.5) and Lemma (2.7) imply  $\xi(N) \leq n(t)\tau(t)e^{-N\rho(t)/n(t)}/\rho(t) + q \log N + (\log |t| + q)/2$ . Since the value of  $N$  that minimizes the expression on the right is large when  $n(t)$  is large, it can be approximated by the root  $N_0$  of the equation  $\tau(t)e^{-N_0\rho(t)/n(t)} = q/N_0$ . If we set  $N_0 = n(t)(\log M)/\rho(t)$  then  $M/\log M = n(t)\tau(t)/q\rho(t)$  so  $M \approx n(t)\tau(t)/q\rho(t) \log[n(t)\tau(t)/q\rho(t)]$  and therefore  $\xi(N_0) \leq q(\log n(t) + \log \log M - \log \rho(t) + 1/2 + 1/\log M) + (\log |t|)/2$ . This inequality and (2.8) gives (2.9).  $\square$

### 3 Two Dimensional Spectral Factorization

Throughout this section  $t \in \mathfrak{T}_2$  with  $t > 0$ . Our objective is to approximate  $t$  by functions  $|p|^2$  where  $p \in \mathfrak{T}_2$ . For  $\sigma > 0$  let  $C_\sigma(\mathbb{T})$  denote the set of functions in  $C(\mathbb{T})$  that have the form  $F(e_1)$  where  $F \in H(\mathbb{A}_\sigma)$ . Then  $C^\omega(\mathbb{T}) = \bigcup_{\sigma>0} C_\sigma(\mathbb{T})$  and  $C^\omega(\mathbb{T}) \otimes \mathfrak{T}_1$  is the subspace of  $\mathfrak{T}_2$  of functions  $f(x, y)$  that are trigonometric polynomials in  $y$  whose coefficients are real analytic functions of  $x$ . For such  $f$  we define  $f_N \in \mathfrak{T}_2$  by

$$f_N(x, y) = \int_{u \in \mathbb{T}} D_N(u) f(x - u, y) du, \quad N \in \mathbb{Z}_+ \quad (3.1)$$

and observe that  $\|f - f_N\|_\infty = \max_{(x,y) \in \mathbb{T}^2} |f(x,y) - f_N(x,y)| \rightarrow 0$  exponentially fast. We now construct a method to compute a spectral factorization  $t = S^-(t)S^+(t)$  where  $S^\pm \in C^\omega(T) \otimes \mathfrak{T}_1$  and  $S^-(t) = \overline{S^+(t)}$ .

**Definition 3.1.** For each  $x \in \mathbb{T}$  define  $h_x \in \mathfrak{T}_1$  by  $h_x(y) = t(x,y)$ ,  $y \in \mathbb{T}$ . Then define  $S^\pm(t) \in C(\mathbb{T}^2)$  by

$$S^\pm(t)(x,y) = (\Psi^\pm h_x)(y). \quad (3.2)$$

Since  $t > 0$  it follows that  $S^\pm(t) \in C^\omega(\mathbb{T}) \otimes \mathfrak{T}_1$  so  $S_N^\pm(t) \rightarrow S^\pm(t)$  at an exponential rate. The remainder of this section computes that rate. Every  $t \in \mathfrak{T}_2$  admits a Fourier expansion  $t = \sum_{(j,k) \in \mathbb{Z}^2} \widehat{t}(j,k) e_{(j,k)}$ , where  $e_{(j,k)}$  are defined by  $e_{(j,k)}(x,y) = e_j(x)e_k(y)$ . We define  $\|\widehat{t}\|_1 = \sum_{(j,k) \in \mathbb{Z}^2} |\widehat{t}(j,k)|$ ,  $n_1(t) = \max\{|j| : \widehat{t}(j,k) \neq 0 \text{ for some } k \in \mathbb{Z}\}$ , and  $n_2(t) = \max\{|k| : \widehat{t}(j,k) \neq 0 \text{ for some } j \in \mathbb{Z}\}$ . We define  $\rho(t) = \min\{1, (\min t)/(2e\|\widehat{t}\|_1)\}$ ,  $\sigma_1(t) = \rho(t)/n_1(t)$ ,  $\tau(t) = \max\{\log(\|t\|_\infty + (\min t)/2), \pi/2 + |\log(\min t/2)|\}$ . The linear maps  $\Gamma_z : \mathfrak{T}_2 \rightarrow \mathfrak{T}_1$   $\Gamma_z e_{j,k} = z^j e_k$ ,  $z \in \mathbb{C} \setminus \{0\}$  are algebra homomorphisms,  $(\Gamma_{e_1(x)} t)(y) = t(x,y)$ ,  $x, y \in \mathbb{T}$ , and for every  $s \in [0, \sigma_1(t)]$  and  $x \in \mathbb{T}$ ,  $\|\Gamma_{e^{se_1(x)}} t - \Gamma_{e_1(x)} t\|_\infty \leq \min t/2$ , and hence for  $z \in \mathbb{A}_{\sigma_1(t)}$

$$\Re \Gamma_z t \geq \min t/2, \quad \text{and} \quad \|\Gamma_z t\|_\infty \leq \|t\|_\infty + \min t/2. \quad (3.3)$$

We observe that if  $z \in \mathbb{A}_{\sigma_1(t)}$  then  $\rho(\Gamma_z t) \geq \rho(t)/2e$  and  $\tau(\Gamma_z t) \leq \tau(t) + \log 2$  and define  $\zeta(t) = \lambda(\rho(t)/2e, \tau(t) + \log 2)$ . Then Theorem (2.9) and the fact that  $n(\Gamma_z t) = n_2(t)$  give

$$\|\Psi^\pm \Gamma_z t\|_\infty < \zeta(t) n_2(t)^{\pi/2}. \quad (3.4)$$

The fact that  $S^\pm(t)(x,y) = (\Psi^\pm \Gamma_{e_1(x)} t)(y)$  and the argument used to prove Lemma (1.1), slightly generalized by considering  $\Psi^\pm \Gamma_z t$  to be a function from  $\mathbb{A}_{\sigma_1(t)}$  with values in the normed subspace  $\mathfrak{T}_1 \subset C(T)$ , give

$$\|S^\pm(t) - S_N^\pm(t)\|_\infty \leq 2\zeta(t) n_2(t)^{\pi/2} \sigma_1(t)^{-1} e^{-N\sigma_1(t)}. \quad (3.5)$$

**Theorem 3.2.** *There exists a function  $\varrho_1 : (0, \infty) \rightarrow (0, \infty)$  such that*

$$N > \varrho_1(\delta) n_1(t) (n_1(t)^\delta - \log \epsilon) \Rightarrow \|S^\pm(t) - S_N^\pm(t)\|_\infty < \epsilon, \quad \epsilon, \delta > 0. \quad (3.6)$$

*Proof.* Define  $N_\epsilon = \frac{n_1(t)}{\rho(t)} [\log(2\zeta(t) n_2(t)^{\pi/2} \sigma_1(t)^{-1}) - \log \epsilon]$ . Then (3.5) implies that  $N \geq N_\epsilon \Rightarrow \|S^\pm(t) - S_N^\pm(t)\|_\infty \leq \epsilon$ . The existence of the function  $\varrho_1$  that satisfies (3.6) then follows from the fact that for  $\epsilon, \delta > 0$   $\lim_{N \rightarrow \infty} N_\epsilon / [n_1(t) (n_1(t)^\delta - \log \epsilon)] = 0$ .  $\square$



## 4 Derivation of Main Result

This section derives the main result in this paper

**Theorem 4.1.** *If  $\alpha$  is rational or if the Liouville-Roth constant  $\mu_0(\alpha) > 2$  then for every  $\beta > 0$  the set  $F_2(\alpha, \beta)$  has property A.*

We recall that for  $\alpha$  irrational  $\mu_0(\alpha)$  is the least upper bound of the set of  $\mu > 0$  for which there exists an infinite number of pairs  $(c, d)$  of relatively prime integers such that

$$|\alpha + c/d| < \frac{1}{|d|^\mu}. \quad (4.1)$$

$\mu_0(\alpha)$  is also called the *irrationality measure* of  $\alpha$  and  $\alpha$  is a Liouville number if  $\mu_0(\alpha) = \infty$ . For  $F \subseteq \mathbb{Z}^2$  define  $F^r = \{(k, j) : (j, k) \in F\}$ . We observe that  $F$  has property A if and only if  $F^r$  has property A. Furthermore, if  $\alpha \neq 0$  then  $F_2^r(\alpha, \beta) = F_2(1/\alpha, \beta/|\alpha|)$ . So without loss of generality we may assume that  $|\alpha| \leq 1$ . We first prove the first assertion for the case that  $\alpha$  is rational by exploiting symmetries of  $\mathbb{Z}^2$ . The modular group  $SL(2, \mathbb{Z}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$  acts as a group of group automorphisms on  $\mathbb{Z}^2$  by  $g(j, k) = (aj + bk, cj + dk)$  and therefore as a group of involution preserving algebra automorphisms on  $\mathfrak{T}_2$  by  $ge_{(j,k)} = e_{g(j,k)}$ . A direct computation gives

$$gF_2(\alpha, \beta) = F_2\left(\frac{c + d\alpha}{a + b\alpha}, \frac{\beta}{|a + b\alpha|}\right), \quad a + b\alpha \neq 0. \quad (4.2)$$

If  $\alpha$  is rational and  $|\alpha| \leq 1$  then there exist coprime integers  $c$  and  $d$  such that  $c + d\alpha = 0$ . Then  $|c| \leq |d|$  and there exist integers  $a$  and  $b$  such that  $|a| \leq |b| \leq |d|/2$  and  $ad - bc = 1$ . We let  $g$  be the corresponding element in  $SL(2, \mathbb{Z})$  and observe that  $gF_2(\alpha, \beta) = F_2(0, \beta|d|)$ . Let  $t \in \mathfrak{T}_2^+(F_2(\alpha, \beta) - F_2(\alpha, \beta))$ . Then  $g(t) \in \mathfrak{T}_2^+(F_2(0, \beta|d|) - F_2(0, \beta|d|))$  so there exists  $k_1, k_2 \in \mathbb{Z}_+$  such that  $k_1 < \beta|d|$ ,  $k_2 < \beta|d|$ , and  $\text{freq}(g(t)) \in \{-k_1 - k_2, \dots, k_1 + k_2\}$ . Then the spectral factor  $S^+(g(t)) \in C^\omega(\mathbb{T}) \otimes \mathfrak{T}_1(\{0, \dots, k_1 + k_2\})$ . Therefore the sequence  $q_N = e_{(0, -k_2)} S_N^+(g(t)) \in \mathbb{T}_2(F_2(0, \beta|d|))$  and the sequence  $p_N = g^{-1}(q_N) \in \mathbb{T}_2(F_2(\alpha, \beta))$  satisfies

$$\|t - |p_N|^2\|_\infty = \|g(t) - |q_N|^2\|_\infty = \|g(t) - |S_N^+(g(t))|^2\|_\infty \rightarrow 0 \quad (4.3)$$

since  $S_N^+(g(t)) \rightarrow S^+(g(t))$  and  $g(t) = |S^+(g(t))|^2$ . This proves the first assertion that  $F_2(\alpha, \beta)$  has property A. We will now prove the second assertion by

using Diophantine approximation of irrational  $\alpha$  by rational numbers. First we need to compute an upper bound for the  $|n_1(p_N)| + |n_2(p_N)|$  when  $N$  is sufficiently large so that approximation error  $\|S^+(t) - S_N^+(t)\|_\infty < \epsilon$ . Theorem (3.2) together with the properties  $g$  and  $g^{-1}$  show that there exists a function  $\varrho_2 : (0, \infty) \rightarrow (0, \infty)$  such that for every  $\epsilon > 0$  there exists  $N \in \mathbb{Z}_+$  such that  $\|S^+(g(t)) - S_N^+(g(t))\|_\infty < \epsilon$  and

$$|n_1(p_N)| + |n_2(p_N)| \leq \varrho_2(\delta) |d|^2 (|d|^\delta - \log \epsilon), \quad \delta > 0. \quad (4.4)$$

Now assume that  $\alpha$  be irrational,  $\mu_0(\alpha) > 2$ ,  $\beta > 0$ , and  $t \in \mathfrak{T}_2^+(F_2(\alpha, \beta) - F_2(\alpha, \beta))$ . Then there exists  $\tilde{\beta} > \beta > 0$  such that  $t \in \mathfrak{T}_2^+(F_2(\alpha, \tilde{\beta}) - F_2(\alpha, \tilde{\beta}))$ . A direct computation shows that for every  $g \in SL(2, \mathbb{Z})$

**Lemma 4.2.**  $(j, k) \in F_2(-c/d, \tilde{\beta})$  and  $|j| < \frac{|d|(\beta - \tilde{\beta})}{|c + d\alpha|} \Rightarrow (j, k) \in F_2(\alpha, \beta)$ .

If  $\mu_0(\alpha) > 2$  there exists  $\mu > 2$  and a sequence of  $g$  with

$$(\beta - \tilde{\beta})/|\alpha + c/d| > (\beta - \tilde{\beta}) |d|^\mu.$$

This sequence grows faster than the expression  $\varrho_2(\delta) |d|^2 (|d|^\delta - \log \epsilon)$  with  $\delta = (\mu - 2)/2$  and this proves the second assertion that  $F_2(\alpha, \beta)$  has property A.

**Remark 4.3.** Does  $F_2(\alpha, \beta)$  have property A if  $\alpha$  is an irrational algebraic number? Since the Thue-Siegel-Roth theorem shows that  $\mu_0(\alpha) = 2$  our method of proof fails.

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